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Two generator subgroups of free products with commuting subgroups

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Abstract

Let H and K be groups with respective subgroups U and V. Let $F=H*K$ be their free product, $R=\{[u,v];u\in U,v\in V\}$ and N be the normal closure of R in F. Then the group $G=F/N$ is called the *free product* of H and K *with commuting subgroups* U and V and is denoted by $H*K/[U,V]$. In the present paper we use geometric techniques of Kapovich and Weidmann to study 2-generator subgroups of free products with commuting subgroups. In the case where the factors are free and J is a two-generator subgroup then either J is free, or $J=\langle a,b|[a^p,b^q]\rangle$ for some $p,q\in\mathbb{N}$ of J is the fundamental group of a graph of cyclic groups where the underlying graph is either homeomorphic to a segment or a circle. © 2002 Published by Elsevier Science B.V.

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1. Introduction

Let H and K be groups with respective subgroups U and V. Let $F=H*K$ be their free product, $R=\{[u,v];u\in U,v\in V\}$ and N be the normal closure of R in F. Then the group $G=F/N$ is called the *free product of* H *and* K *with commuting subgroups* U and V. Such groups are natural intermediate constructions between free products (where U and V are trivial) and direct products (where $U=H$, $V=K$), and arise in many different contexts. In particular, *graph groups* (see [2] or [5]) are a type

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of iterated free product with commuting subgroup construction. Graph groups arise in topological contexts and have been shown to be automatic [5]. The automaticity of other more general free products with commuting subgroups can also be established.

Miller and Schupp [10] have studied the geometry of free products with commuting subgroups and showed that the conditions under which the word problem is solvable are the same as in the case of free products with amalgamation. Hurwitz [3], also using geometric techniques proved that if H and K are free groups and U and V are finitely generated then the group G has solvable conjugacy problem.

Free products with commuting subgroups can be expressed as an iterated free product with amalgamation (see [9]), namely

$$H * V/[U,V] = H *_U (U \times V) *_V K.$$

Because of this, the analysis of subgroups of such groups can be handled by the generalized Nielsen reduction techniques developed for free products with amalgamation and HNN groups (see [14,11]). Geometric versions of these techniques have been developed by Kapovich and Weidmann [6] for 2-generator groups and by Weidmann [13] in general.

In this paper, we use the geometric techniques of Kapovich and Weidmann to describe 2-generator subgroups of free products with commuting subgroups. Note that in the following we consider a cyclic group to be a free product of cyclic groups.

Theorem 1. *Let H and K be groups, $U < H$ and $V < K$ be two subgroups and $G = H * K/[U,V] = H *_U (U \times V) *_V K$. Let further J be a subgroup of G generated by g and h and suppose that J is not a free product of cyclic groups. Then one of the following holds*:

(1) $wJw^{-1} \subset H *_U (U \times V)$ *for some* $w \in G$,

(2) $wJw^{-1} \subset (U \times V) *_V K$ *for some* $w \in G$,

(3) $\{g,h\}$ *is Nielsen-equivalent to* $\{f,s\}$ *such that J has the presentation*

$$\langle f,s \mid f^p, s^q, [f^{n_1}, s^{n_2}]\rangle$$

for some $n_1, n_2 \in \mathbb{N}$ *and* $p,q \in \mathbb{N} \cup \{\infty\}$,

(4) $\{g,h\}$ *is Nielsen-equivalent to* $\{f,s\}$ *such that J has the presentation*

$$\langle f,s \mid f^p, [f^{n_1}, s f^{n_2} s^{-1}]\rangle$$

for some $n_1, n_2 \in \mathbb{N}$ *and* $p = lcm(n_1, n_2)$.

If both H and K are torsion-free this result then yields the following since p and q cannot be finite:

Corollary 2. *Let H and K be torsion-free groups with subgroups U and V and $G = H * K/[U,V] = H *_U (U \times V) *_V K$ and $J = \langle g,h\rangle < G$ be a non-free subgroup. Then one of the following holds*:

(1) $wJw^{-1} \subset H *_U (U \times V)$ *for some* $w \in G$,

(2) $wJw^{-1} \subset (U \times V) *_V K$ *for some* $w \in G$,

(3) $\{g,h\}$ *is Nielsen-equivalent to* $\{f,s\}$ *such that J has the presentation* $\langle f,s \mid [f^{n_1}, s^{n_2}]\rangle$ *for some* $n_1, n_2 \in \mathbb{N}$.

In the case of a free product with commuting subgroups of type $F_n * F_m/[U,V]$ where F_n and F_m are free groups and $U < F_n$ and $V < F_m$ more can be said. We prove the following:

Theorem 3. *Let $G = F_n * F_m/[U,V]$ and J be a two-generator subgroup of G. Then either J is free, or $J = \langle a,b | [a^p, b^q] \rangle$ for some $p,q \in \mathbb{N} \cup \{\infty\}$ or J is the fundamental group of a graph of cyclic groups where the underlying graph is either homeomorphic to a segment or a circle.*

Recall that a subgroup U of a group G is *isolated* if it contains all roots of elements in U, that is $g^n \in U$ for some $n > 1$ implies $g \in U$. In Theorem 3 if both U and V are isolated subgroups of the respective free groups F_n, F_m then we get the following:

Theorem 4. *Let $G = F_n * F_m/[U,V]$ with U and V isolated subgroups of F_n, F_m, respectively. Then any two-generator subgroup of G is either free or free abelian.*

In order to prove the main theorems, a given generating set of a subgroup will be replaced by a new generating set, from which we can read off the needed information. In Section 2, we describe situations where the induced splitting of a subgroup of a fundamental group of a graph of groups can be read of a generating set. In Sections 3 and 4 we will prove Theorems 1 and 3.

2. The induced splitting of a subgroup

Let G be the fundamental group of a graph of groups A and

$$G \times T \to T, \ (g,x) \mapsto gx$$

be the action of G on the Bass–Serre tree associated to the splitting A. Let further $U < G$ be a subgroup. Since U also acts on T, we can see, after replacing T by a U-minimal subtree $T_U \subset T$ that U is itself the fundamental group of a graph of groups where all vertex (edge) groups are subgroups of conjugates of the vertex (edge) groups of A. This has been first observed by Karrass and Solitar for amalgamated products and HNN-extensions [7,8]. This splitting of U is called the *induced* splitting. For details on the Bass–Serre theory we refer the reader to the book of Serre [12]. Note that the tree T_U is unique unless U acts trivially, i.e. unless U has a fixed point. Therefore we can usually speak about *the* minimal subtree.

Given a subgroup $U < G$ it is in general difficult to determine the minimal U-invariant subtree T_U. In this section however we describe a situation in which we can easily do this.

We assume familiarity with the Bass–Serre theory and only establish the notation for a graph of groups:

A *graph* Γ consists of a set $V\Gamma$ of vertices, a set $E\Gamma$ of (oriented) edges and two maps $\alpha : E\Gamma \to V\Gamma$ and $\omega : E\Gamma \to V\Gamma$ that assign every edge its initial and its terminal vertex.

A *graph of groups A* with underlying graph Γ is a quadruple

$$(\Gamma, \{H_v \mid v \in V\Gamma\}, \{H_e \mid e \in E\Gamma\}, \{\alpha_e, \omega_e \mid e \in E\Gamma\})$$

where the H_v and H_e are the vertex and edge groups and the maps $\alpha_e : H_e \to H_{\alpha(e)}$ and $\omega_e : H_e \to H_{\omega(e)}$ are the boundary monomorphisms.

Let now G be a group, T a simplicial tree and $G \times T \to T$, $(g,x) \mapsto gx$, be a simplicial action without inversion. Let further U be a subgroup of G. We first look at some examples where we can immediately see what the induced splitting of U is.

(1) Suppose that U fixes a vertex $v \in T$, i.e. that U is an elliptic subgroup. It is clear that v is a minimal U-invariant subtree and that the induced splitting is a graph of groups that consists of a single vertex with vertex group U.

(2) Suppose that there exists an edge $e = [x_1, x_2]$ of T and groups $G_1, G_2 < G$ where $G_i x_i = x_i$ for $i = 1, 2$, $C = G_1 \cap \operatorname{Stab} e = G_2 \cap \operatorname{Stab} e$ and neither G_1 nor G_2 fix e. Then $U = \langle G_1 \cup G_2 \rangle = G_1 *_C G_2$. This is clear since the Bass–Serre tree associated to the amalgamated product $G_1 *_C G_2$ can clearly be embedded in T such that the vertices fixed under the action of G_1 and G_2 coincide with x_1 and x_2. To see this, we only need to remark that two representatives g and h of distinct cosets of G_i modulo C map e to different edges emanating at x_i, this however is trivial since otherwise $gh^{-1} \in \operatorname{Stab} e \cap G_i \subset C$ which is a contradiction. Together this implies that $U = G_1 *_C G_2$ and that there is a U-equivariant isomorphism between the Bass–Serre tree of $U = G_1 *_C G_2$ and Ue which is the minimal U-invariant subtree of T.

(3) Suppose that there exists an edge $e = [x_1, x_2]$ of T, a group $H < G$ such that $Hx_1 = x_1$ and an element $t \in G$ such that $tx_1 = x_2$ and that $C = H \cap \operatorname{Stab} e = tHt^{-1} \cap \operatorname{Stab} e$. Then $U = \langle H \cup \{t\} \rangle = H*_C$ is the HNN-extension of H where the stable letter t conjugates C to $t^{-1}Ct$. Again there is a U-equivariant isomorphism between the Bass–Serre tree of $U = H*_C$ and Ue which is the minimal U-invariant subtree of T.

Using the same arguments we can describe situations where we can, for a given subgroup U, determine the minimal U-invariant subtree of T and therefore the induced splitting. For a tree T we will, as for graphs, denote the set of vertices by VT and the set of edges by ET.

Suppose that there are two subtrees T_1 and T_2 of T, that $T_1 \subset T_2$ and that every vertex $v \in VT_2 - VT_1$ can be joined to a vertex $w_v \in VT_1$ by a single edge $e_v = [v, w_v]$. We say that a subgroup $U \subset G$ is *controlled by a tuple*

$$Y(U) = (T_1, T_2\{G_v \mid v \in VT_2\}, \{t_v \mid v \in VT_2 - VT_1\})$$

if the following are fulfilled:

(1) $G_v \subset G$ for all $v \in VT_2$ and $t_v \in G$ for all $v \in VT_2 - VT_1$.
(2) $U = \langle (\bigcup_{v \in VT_1} G_v) \cup \{t_v \mid v \in VT_2 - VT_1\} \rangle$.
(3) $G_v v = v$ for all $v \in VT_2$, i.e. G_v is elliptic with fixed point v.
(4) $t_v T_1 \cap T_1 = \emptyset$ and $t_v T_1 \cap T_2 = v$ for all $v \in VT_2 - VT_1$.
(5) $G_v = t_v G_{t_v^{-1} v} t_v^{-1}$ for all $v \in VT_2 - VT_1$.
(6) $G_{v_1} \cap \operatorname{Stab} e = G_{v_2} \cap \operatorname{Stab} e$ for every edge $e = [v_1, v_2]$ of T_2.
(7) $t_v^{-1} e_v \neq t_{\bar{v}}^{-1} e_{\bar{v}}$ for all v, $\bar{v} \in VT_2 - VT_1$ and $v \neq \bar{v}$.

(8) No two edges of T_3 emanating at a vertex $v \in VT_1$ are G_v-equivalent where $T_3 = T_2 \cup (\bigcup_{\{v \in VT_2 - VT_1\}} \{t_v^{-1} e_v\})$.

Note that condition (4) clearly implies that $t_v^{-1} v \in VT_1$. If we further have that there exists no vertex $x \in VT_1$ such that a component C of $T_1 - \{x\}$ is also a component of $T_3 - \{x\}$ and that $G_v \subset G_x$ for all $v \in VC$ then we say that $Y(U)$ *controls* U *minimally*.

The groups G_v will usually be described by generating sets S_v. It is clear that the subgroup U is implicit in the tuple $Y(U)$. For every subgroup U that is controlled by a tuple $Y(U) = (T_1, T_2, \{G_v \mid v \in VT_2\}, \{t_v \mid v \in VT_2 - VT_1\})$ we define the *associated graph of groups*

$$A_{Y(U)} = (\Gamma, \{G_v \mid v \in V\Gamma\}, \{G_e \mid e \in E\Gamma\}, \{\alpha_e, \omega_e \mid e \in E\Gamma\}).$$

We first define the underlying graph Γ to be the graph obtained from T_2 by identifying v with $t_v^{-1} v$ for every $v \in VT_2 - VT_1$. This means we have $V\Gamma = VT_1$ and $E\Gamma = ET_2$ and the maps $\alpha : E\Gamma \to V\Gamma$ and $\omega : E\Gamma \to V\Gamma$ are chosen in the obvious way for $e \in ET_1 \subset E\Gamma$ and for the edges $e_v = [v, w_v]$ with $v \in VT_2 - VT_1$ (remember that $w_v \in VT_1 = V\Gamma$ and $v \in VT_2 - VT_1$, i.e. $v \notin V\Gamma$) we define $\alpha(e_v) = w_v$ and $\omega(e_v) = t_v^{-1} v \in VT_1 = V\Gamma$. It is clear that T_1 can be considered as a maximal subtree of Γ.

We define the vertex group of every vertex v to be the group G_v and the edges group of an edge $e \subset \Gamma$ to be $\operatorname{Stab} e \cap G_{v_1} = \operatorname{Stab} e \cap G_{v_2}$ where $e = [v_1, v_2]$. We further define the boundary monomorphisms $\alpha_e : G_e \to G_{\alpha(e)}$ for all $e \in E\Gamma$ and $\omega_e : G_e \to G_{\omega(e)}$ for all $e \in ET_1 \subset E\Gamma$ to be the inclusion map and define $\omega_e : G_e \to G_{\omega(e)}$ by $\omega(g) = t_v^{-1} g t_v$ if $e = e_v \in E\Gamma - ET_1$. This means we have $\omega_{e_v} : G_{e_v} \to G_{t_v^{-1} v}$ is defined by $\omega(g) = t_v^{-1} g t_v$.

Let now $\bar{U} = \pi_1(A_{Y(U)}, T_1)$, $\bar{T}$ be the Bass–Serre tree corresponding to the graph of groups $A_{Y(U)}$ and $\bar{U} \times \bar{T} \to \bar{T}$ be the corresponding action.

Condition (1)–(5) imply that there exists a homomorphism $\varphi : \bar{U} \to G$ that extends the identity map on the groups G_v and maps the stable letters of $\bar{U} = \pi_1(A_{Y(U)}, T_1)$ onto the elements t_v. We clearly have $\varphi(\bar{U}) = U$. Equivariant extension further yields a map $f : \bar{T} \to T$ such that the diagram

$$\begin{array}{ccc} \bar{U} \times \bar{T} & \longrightarrow & \bar{T} \\ \Big\downarrow{\scriptstyle (\varphi, f)} & & \Big\downarrow{\scriptstyle f} \\ G \times T & \longrightarrow & T \end{array}$$

is commutative. Note that φ is injective when restricted to any vertex stabilizer of $\bar{T}$. Conditions (6)–(8) further guarantee that the map f is locally injective which implies that the pair (φ, f) is injective. If $Y(U)$ controls U minimally we also get that the action $\bar{U} \times \bar{T} \to \bar{T}$ is minimal. If we restrict the action of U on T to the minimal invariant subtree T_U we therefore get

$$\begin{array}{ccc} \bar{U} \times \bar{T} & \longrightarrow & \bar{T} \\ \Big\downarrow{\scriptstyle (\varphi, f)} & & \Big\downarrow{\scriptstyle f} \\ U \times T_U & \longrightarrow & T_U, \end{array}$$

where (φ, f) is an isomorphism of group actions. In particular this implies that $\bar{U} \cong U$. In order to get this isomorphism we clearly do not need that $Y(U)$ controls U minimally, i.e. we get the following:

Proposition 5. *Let $U \subset G$ be a subgroup controlled by the tuple $Y(U)$ and $A_{Y(U)}$ be the associated graph of groups. Then $U \cong \pi_1(A_U)$.*

A more formal proof of the above could be given along the lines of the discussion in [1, pp. 204–211].

3. Two-generated subgroups of $H *_U (U \times V) *_V K$

Let H and K be two groups and let $U < H$ and $V < K$ be two subgroups. We study two-generator subgroups of the group $G = H * K/[U,V]$. It is easy to see that G can be written as the iterated amalgamated product

$$H *_U (U \times V) *_V K,$$

i.e. can also be written as the fundamental group of a graph of groups with vertex groups H, K and $U \times V$ and edge groups U and V where the boundary monomorphisms are the identity on U and V. We will study 2-generator subgroups of G by studying the action of G on the Bass–Serre tree corresponding to this splitting. We call all edges that are G-equivalent to the edge fixed by U *red* and all edges that are G-equivalent to the edge fixed by V *blue*. Since U and V are *conjugacy separated* in $U \times V$, i.e. $gUg^{-1} \cap V = 1$ for all $g \in U \times V$, it follows that no non-trivial element of G fixes a blue and a red edge. For any $g \in G$ we denote by T_g the subtree of T that contains all points of T that are fixed under the action of some non-trivial power of g. Suppose now that x is fixed under the action of g, this implies in particular that $x \in T_g$. Suppose further that $y \in T_g$. Since the power of g that fixes y also fixes x, this power also fixes the segment $[x, y]$ which implies that it either only contains red edges or only blue edges. We have shown that T_g contains at most one point, namely x, that is adjacent to a blue and a red edge, i.e. every component of $T_f - \{x\}$ is either all blue or all red. The following theorem of [6] is central to our investigation:

Theorem 6. *Let G be a group acting on a simplicial tree T without inversions. Suppose that $g, h \in G$ such that $\langle g, h\rangle$ is not a free product of cyclic groups. Then $\{g, h\}$ is Nielsen equivalent to $\{f, s\}$ such that either*
(1) $T_f \cap T_s \neq \emptyset$ *or*
(2) $T_f \cap sT_f \neq \emptyset$.

We proceed with the proof of Theorem 1:

Proof of Theorem 1. Suppose that J is not a free product of cyclic groups. By Theorem 6 it suffices, after replacing $\{g, h\}$ with the Nielsen equivalent pair $\{f, s\}$, to investigate the cases where $T_f \cap T_s \neq \emptyset$ and $T_f \cap sT_f \neq \emptyset$:

Case 1 ($T_f \cap T_s \neq \emptyset$): Choose $z \in T_f \cap T_s$ and $x \in T_f$ and $y \in T_s$ such that $fx = x$ and $sy = y$. By our discussion above the segments $[x,z]$ and $[y,z]$ each contain only blue or only red edges. If $[x,z]$ and $[y,z]$ contain no edges, i.e. if $[x,z] = [y,z] = z$, then $Uz = z$ and U is conjugate to a subgroup H, K or $U \times V$. We can therefore assume that either $[x,z]$ or $[y,z]$ contains at least one edge.

Suppose now that $[x,z]$ and $[y,z]$ contain only red edges. After conjugation we can assume that z is fixed under the action of either H or $U \times V$ since any vertex incident to a red edge is G-equivalent to one of these two vertices and since conjugation preserves colours. Now f and s both fix vertices that can be reached from z by all-red segments. This clearly implies that $f, s \in H *_U (U \times V)$ which proves the assertion. In the case that $[x,z]$ and $[y,z]$ only contain blue edges the same arguments show that U is conjugate to a subgroup of $(U \times V) *_V K$.

Suppose now that $[x,z]$ contains only red edges, that $[y,z]$ contains only blue edges and that both trees contain at least one edge. Now $[x,z] \cap [y,z]$ must consist of exactly one vertex, namely z, since there are clearly no two distinct vertices that are joined by an all-red and by an all-blue path. After conjugation we can assume that z is fixed under the action of $U \times V$ since every vertex of T that is incident to a blue and to a red edge is G-equivalent to the vertex fixed by $U \times V$. By the same reasoning as before we get that $f \in H *_U (U \times V)$ and $s \in (U \times V) *_V K$. Since powers of f and s fix z and since $\operatorname{Stab} z = U \times V$ we can choose $n_1, n_2 \in \mathbb{N}$ minimal such that $f^{n_1} \in U \times V$ and $s^{n_2} \in U \times V$. If $n_1 = 1$ (the case $n_2 = 1$ is analogous) we get that $s, t \in (U \times V) *_V K$ which proves the assertion, i.e. we are left with the case $n_1, n_2 \geqslant 2$.

Since f^{n_1} fixes x and z it also fixes the edge of $[x,z]$ emanating at z which is stabilized under U, i.e. $f^{n_1} \in U$. Analogously we see that $s^{n_2} \in V$. This implies in particular that $\langle f^{n_1}, s^{n_2} \rangle = \langle f^{n_1} \rangle \times \langle s^{n_2} \rangle$ and that $\langle f^{n_1}, s^{n_2} \rangle \cap U = \langle f^{n_1} \rangle$ and $\langle f^{n_1}, s^{n_2} \rangle \cap V = \langle s^{n_2} \rangle$. We assign to $J = \langle f, s \rangle$ a tuple $Y(J) = (T_1, T_2 = T_1, \{G_v \mid v \in VT_1\}, \emptyset)$ that controls J. Consequently we are able to see that the presentation of $\pi_1(A_{Y(J)})$ and therefore also of J is $\langle f, s \mid f^p, s^q, [f^{n_1}, s^{n_2}] \rangle$. The tree $T_1 = T_2$ consists of the segment $[x, y] = [x,z] \cup [y,z]$. We further define $G_z = \langle f_z = f^{n_1}, s_z = s^{n_2} \rangle$ and $G_v = \langle f^{n_v} \rangle$ if $v \in [x,z)$ where f^{n_v} is the smallest power of f fixing v. Analogously we define $G_v = \langle s^{n_v} \rangle$ if $v \in [y,z)$ where s^{n_v} is the smallest power of s fixing v. It is clear that $J = \langle \bigcup_{v \in VT_1} G_v \rangle$ since $\{f, s\} \subset G_x \cup G_y$ and that $G_v \subset J$ for all $v \in VT_1$. It is furthermore easy to see that the tuple $Y(J)$ has all other properties to control J. By Proposition 5 this implies that J is isomorphic to the fundamental group of the graph of groups corresponding to the described controlled set. After collapsing all edges not incident to z we are left with a graph of groups that has three vertices with vertex groups $\langle f \mid f^p \rangle$, $\langle f_z, s_z \mid f_z^{p/n_1}, s_z^{q/n_2}, [f_z, s_z] \rangle$, $\langle s \mid s^q \rangle$ where p and q are the orders of f and s and edge groups $\langle f_z = f^{n_1} \rangle$ and $\langle s_z = s^{n_2} \rangle$. This clearly implies that $J = \langle f, s \rangle = \langle f, s \mid f^p, s^q, [f^{n_1}, s^{n_2}] \rangle$.

Case 2 ($T_f \cap sT_f \neq \emptyset$): Choose $z \in T_f$ such that $fz = z$ and further $x, y \in T_f$ such that $sx = y$. Such x and y exist since $T_f \cap sT_f \neq \emptyset$. Recall that $[x,z]$ and $[y,z]$ must each be all red or all blue.

Suppose now that $[x,z] \cup [y,z]$ only contains red edges (the case that $[x,z] \cup [y,z]$ contains only blue edges is analogous). After conjugation we can assume that the edge e_U fixed under the action of U lies in T_f and contains z, in particular $f \in H *_U (U \times V)$. Since the segment $[x, y]$ contains only red edges and can be joined to e_U by red edges only it follows that also $s \in H *_U (U \times V)$. It follows that $J = \langle f, s\rangle \subset H *_U (U \times V)$ which proves the assertion.

Suppose now that $[x,z]$ and $[y,z]$ each contain at least one edge and that $[x,z]$ contains only red edges and $[y,z]$ contains only blue edges, the opposite case is analogous. This clearly implies that $[x,y]=[x,z]\cup[z,y]$. Let $e_1=[x,x']$ and $e_2=[y,y']$ be the edge of $[x,z]$ and $[y,z]$ containing x and y, respectively. Now choose n_1 and n_2 minimal such that $f^{n_1}e_1 = e_1$ and $f^{n_2}e_2 = e_2$. Note that this can only happen if f is of finite order since otherwise $f^{lcm(n_1,n_2)}$ would be non-trivial and fix a red and a blue edge which is impossible. We denote the order of f by p. Since $f^{lcm(n_1,n_2)}$ must be trivial we get that $p|lcm(n_1,n_2)$. Since also $n_1|p$ and $n_2|p$ it follows that $p = lcm(n_1,n_2)$. Now $sf^{n_1}s^{-1}$ fixes the red edge $se_1 = s[x,x'] = [sx,sx'] = [y,sx']$. In particular we get that the subgroup $\langle f^{n_2}, sf^{n_1}s^{-1}\rangle$ fixes y. As in the first case it is clear the group $G_y = \langle sf^{n_1}s^{-1}\rangle \times \langle s^{n_2}\rangle$ and that $G_y \cap \operatorname{Stab} e_2 = \langle f^{n_1}\rangle$ and that $G_y \cap \operatorname{Stab} se_1 = \langle sf^{n_1}s^{-1}\rangle$. Again we assign to $J=\langle f,s\rangle$ a tuple $Y(J)=(T_1,T_2,\{G_v \mid v\in VT_2\},\{t_v \mid v \in VT_2 - VT_1\})$ that controls J. We put $T_2 = [x,z]\cup[y,z] = [x,y]$, $T_1 = [x,y']$, $G_v = \langle f\rangle \cap \operatorname{Stab} v$ for $v \notin \{x,y\}$, $G_x = \langle f^{n_1}, s^{-1}f^{n_2}s\rangle$, $G_y = \langle sf^{n_1}s^{-1}, f^{n_2}\rangle$ and $t_y = s$. It is easily checked that this tuple controls J. After collapsing all edges in the corresponding graph of groups except the edges associated to e_1 and e_2 we obtain a graph of groups whose underlying graph consists of two vertices x and z and who has two edges e and f joining the two vertices. The vertex groups are $G_z = \langle f | f^p\rangle$ and $G_x = \langle a,b | [a,b], a^{p/n_1}, b^{p/n_2}\rangle$ and the edge groups $G_e = \langle a | a^{p/n_1}\rangle$ and $G_f = \langle b | b^{p/n_2}\rangle$. The boundary monomorphisms into G_x are the identity and into G_z map a to f^{n_1} and b to f^{n_2}. This clearly shows that $\pi_1(A_{Y(J)})$ and therefore also J has the presentation $\langle f,s | f^p, [f^{n_1}, sf^{n_2}s^{-1}]\rangle$. $\square$

4. Two-generated subgroups of $F_n * F_m/[U, V]$

We now restrict ourselves to the case where $H = F_n$, a free group of rank n and $K = F_m$, a free group of rank m and complete the proof of Theorem 3. In view of Corollary 2 we only need to study the case that J is a non-free 2-generated subgroup of $G = F_n *_U (U \times V)$, the case that J is a subgroup of $(U \times V) *_V F_m$ is analogous. We first prove the following:

Lemma 7. *Let $G = F_n *_U (U \times V)$. Let J be a non-free 2-generated subgroup of G. Then the intersection of J with every conjugate of F_n in G is either trivial or cyclic.*

Proof. Let

$$\phi : H \to F_n$$

be the surjective homomorphism that quotients out the normal closure of V. It is clear that ϕ is an isomorphism when restricted to a conjugate of F_n (in H). Since $\phi(J)$ is a (free) subgroup of F_n and since J is assumed to be non-free it follows that $\phi(J)$ is either trivial or cyclic since J cannot have a free group of rank 2 or higher as a homomorphic image. $\square$

Another simple tool is the following:

Lemma 8. *Let G be a group acting on a simplicial tree T and suppose that $e=[x,y]$ and $f=[x,z]$ are two distinct edges emanating at x. Suppose further that g fixes x, that g^p fixes e and g^q fixes f where $(p,q)=1$. Then $g^k f \neq e$ for all $k \in \mathbb{Z}$.*

Proof. Assume that $g^k f = e$ for some $k \in \mathbb{Z}$. This implies that $g^p f = g^{-k} g^p g^k f = g^{-k} g^p e = g^{-k} e = f$, i.e. g^q and g^p fix f. It follows that g fixes f since $g \in \langle g^p, g^q \rangle$ and therefore $g^k f \neq e$, a contradiction. □

The proof of Theorem 3 relies on the following two lemmas:

Lemma 9. *Let $G = F_n *_U (U \times V)$ with V free. Suppose further that $f,s \in G$ such that $T_f \subset T_s \neq \emptyset$. Then $J = \langle f,s \rangle$ is either free or the fundamental group of a graph of cyclic groups where the underlying graph is homeomorphic to a segment or has (possibly after exchanging f and s) the presentation $\langle f,s | [f,s^n] \rangle$ for some $n \in \mathbb{N}$.*

Proof. If $T_f = T_s$ consists of a single vertex, then after conjugation we get that either $J \subset F_n$ which implies that J is free or that $J \subset U \times V$. Since U and V are free groups and J is two-generator this latter case implies that either J is free or free abelian. To see this suppose that $J = \langle x, y \rangle$ with $x = (u_1, v_1)$, $y = (u_2, v_2)$ and $u, u_2 \in U, v_1, v_2 \in V$. A non-trivial relation in x, y would imply a non-trivial relation in u_1, u_2 which is possible only if u_1, u_2 are in the same cyclic subgroup of U. Analogously for v_1, v_2. Therefore, if J is not free we must have $x = (u^\alpha, v^\beta)$, $y = (u^\gamma, v^\delta)$ for some $u \in U$, $v \in V$ and integers $\alpha, \beta, \gamma, \delta$ and hence J is free abelian.

It follows that we can assume that either T_f or T_s does not consist of a single vertex. Without loss of generality we assume that T_s contains an edge. This implies that s is conjugate to an element of F_n since no power of an element of $U \times V - U$ lies in U, i.e. no element that is conjugate to an element of $U \times V$ but not to an element of U (and therefore F_n) has a power that fixes an edge. We look at two situations and it is clear that they cover all cases modulo conjugation.

(1) $f \in U \times V - U$, i.e. $T_f = \{x\}$, where x is the vertex fixed under the action of $U \times V$. Suppose that y is fixed under the action of s. We define a tuple $Y(J) = (T_1, T_2, \{G_v \mid v \in VT_1\}, \emptyset)$ such that J is controlled by $Y(J)$. We put $T_1 = T_2 = [x, y]$ and $G_v = \mathrm{Stab}\, v \cap \langle s \rangle$ for $v \neq x$ and $G_x = \langle f, s^n \rangle$ where s^n is the smallest power of s fixing x. The only non-trivial assertion we have to verify in order to see that $Y(J)$ controls J is that $\mathrm{Stab}\, e \cap G_x = \mathrm{Stab}\, e \cap G_z$ where $e = [x,z]$ is the edge of $[x, y]$ containing x. It is clear that $\langle s^n \rangle = \mathrm{Stab}\, e \cap G_z$, lies in U. Since $f \in U \times V - U$ it follows that $f = v + u$ with $v \in V - 1$ and $u \in U$.

If $[u, s^n] = 1$ then f and s^n clearly generate a free abelian group of rank 2 and $\mathrm{Stab}\, e \cap G_x = \mathrm{Stab}\, e \cap \langle f, s^n \rangle = \langle s^n \rangle$ which implies that $Y(J)$ controls J. Collapsing all edges but the edge corresponding to e we get that $\langle s, f \rangle = \langle f, s^n \rangle *_{\langle s^n \rangle} \langle s \rangle$, i.e. that $J = \langle f, s | [f, s^n] \rangle$.

If $[u, s^n] \neq 1$ then $s^n = \phi(s^n)$ and $u = \phi(f)$ (ϕ as chosen above) do not commute, i.e. do not lie in a cyclic subgroup of F_n, which implies that $\langle f, s \rangle$ is free by Lemma 7.

(2) f is conjugate to an element of F_n. So is s by the remark above. Choose two vertices x and y that are G-equivalent to the vertex fixed under the action of F_n such that $fx=x$ and $sy=y$. We define a tuple $Y(J)=(T_1,T_2,\{G_v\,|\,v\in VT_1\},\emptyset)$ such that $Y(J)$ controls J. We put $T_1=T_2=[x,y]$ and replace f and s by generators of the maximal cyclic subgroups of J stabilizing x and y, respectively. We define $G_x=\langle f\rangle, G_y=\langle s\rangle$ and for $v\in(x,y)$ define $G_v=\langle J\cap \mathrm{Stab}\, e, J\cap \mathrm{Stab}\, f\rangle$ where $e_v=[a,v]$ and $f_v=[b,v]$ are the two edges of $[x,y]$ containing v. Since all of these groups are conjugate to a subgroup of F_n or of U (and therefore of F_n), they are all cyclic and the edges e_v and f_v are not G_v-equivalent by Lemma 8. This implies that $Y(J)$ controls J and therefore the assertion of the Lemma holds. $\square$

Lemma 10. *Let $G=F_n *_U (U\times V)$ with V free. Suppose further that $f,s\in G$ such that sf^z acts fixed point free for all $z\in\mathbb{Z}$ and that $T_f \subset sT_f\neq\emptyset$. Then J is the fundamental group of a graph of cyclic groups where the underlying graph is homeomorphic to a circle.*

Proof. We first replace s with the element of $\{sf^z\,|\,z\in\mathbb{Z}\}$ that has minimal translation length and denote it again by s. This does not change the fact that $T_f\cap sT_f\neq\emptyset$ since f^z maps T_f to T_f. By assumption this translation length is greater than 0. We choose $x\in T_f$ such that $fx=x$ and p to be the projection of x onto the axis of s and define $q:=sp$. Since $[p,q]$ lies on the axis of s we get that $[x,p]\cap[p,q]=p$. Let f^n be the smallest power of f fixing p. We define a tuple $Y(J)=(T_1,T_2,\{G_v\,|\,v\in VT_2\},\{t_v\,|\,v\in VT_2-VT_1\})$ such that $Y(J)$ controls $J=\langle f,s\rangle$. The items are defined the following way:

$T_1=[x,p]\cup[p,r_q]$. $T_2=[x,p]\cup[p,q]$ where r_q is the vertex of $[p,q]$ adjacent to q. $G_v=\mathrm{Stab}\, v\cap\langle f\rangle$ for $v\in[x,p]$, in particular $G_p=\langle f^n\rangle$. We further define $G_q=\langle sf^ns^{-1}\rangle$ and $G_v=\langle \mathrm{Stab}\, e_v\cap\langle f^n,sf^ns^{-1}\rangle, \mathrm{Stab}\, f_v\cap\langle f^n,sf^ns^{-1}\rangle\rangle$ for $v\in(p,q)$ where e_v and f_v are the two edges of $[p,q]$ that are adjacent to v. By Lemma 7 all of the vertex groups are cyclic since they are either conjugate to subgroups of F_n or of U (and therefore also of F_n). We further define $t_q:=s^{-1}$. We next show that $G_p=\langle f^n,sf^ns^{-1}\rangle\cap \mathrm{Stab}\, p$. The first inclusion is trivial since $G_p=\langle f^n\rangle$ and therefore $G_p\subset\langle f^n,sf^ns^{-1}\rangle\cap\mathrm{Stab}\, p$. To see the second inclusion we first observe that $\langle f^n,sf^ns^{-1}\rangle\cap\mathrm{Stab}\, p$ is conjugate to a subgroup of F_n and therefore cyclic by Lemma 7. This holds since $\langle f^n,sf^ns^{-1}\rangle\cap\mathrm{Stab}\, p$ lies in a kernel of the quotient map $G\to G/N(F_n)$ which is the identity when restricted to V. If $\langle f^n,sf^ns^{-1}\rangle\cap\mathrm{Stab}\, p\not\subset G_p$ then there exists a root of f^n in $\langle f^n,sf^ns^{-1}\rangle\cap \mathrm{Stab}\, p$. This however is impossible since no root of f^n lies in the normal closure $N_{F_n}(f^n)$ which equals $F_n\cap N_G(f^n)$. The first part of this statement follows from [4], the second part can be seen by quotienting out the normal closure of V. Analogously we see that $G_q=\langle f^n,sf^ns^{-1}\rangle\cap\mathrm{Stab}\, q$. It further follows from Lemma 7 that e_v and f_v are not G_v-equivalent for $v\in(p,q)$. In order to see that $Y(J)$ controls J it remains to be shown that no two edges of T_3 emanating at p are G_p-equivalent. If $x=p$ then there are only two edges, namely the edge $[r_p,p]$ where r_p is the edge of $[p,q]$ that is adjacent to p and the edge $s^{-1}[q,r_q]=[s^{-1}q,s^{-1}r_q]=[p,s^{-1}r_q]$. These two edges are not G_p-equivalent because of the minimality of s. If $p\neq x$ then there is another edge emanating at p, namely the edge $[x_p,p]\subset[x,p]$. This edge however is fixed by f^n and is therefore not G_p-equivalent to the other two. Collapsing all edges in the corresponding graph of groups that correspond to an edge of $[x, p]$ yields a graph of cyclic groups with underlying graph homeomorphic to a circle. $\square$

Proof of Theorem 3. We suppose that J is a 2-generated non-free subgroup of $F_n * F_m/[U,V]$ that is not of type $\langle a,b|[a^p,b^q]\rangle$ for some $p,q\in\mathbb{N}\cup\{\infty\}$. It follows from Corollary 2 that after conjugation we have that either $J\subset F_n *_U (U\times V)$ or that $J\subset (U\times V)*_V F_m$. We assume that $J\subset F_n *_U (U\times V)$, the other case is analogous. We study the action of J on the Bass–Serre tree corresponding to the amalgamated product $F_n *_U (U\times V)$.

By Theorem 6 we can assume that J is generated by elements f and s such that either $T_f\cap T_s\neq\emptyset$ or that $T_f\cap sT_f\neq\emptyset$. If $T_f\cap T_s\neq\emptyset$ then the assertion of Theorem 4 follows directly from Lemma 9. If $T_f\cap sT_f\neq\emptyset$ and sf^z acts without fixed point for all $z\in\mathbb{Z}$ then the assertion follows from Lemma 10. If $T_f\cap sT_f\neq\emptyset$ and sf^z acts with a fixed point for some $z\in\mathbb{Z}$ then J is generated by the elliptic elements f and $s'=sf^z$. If $T_f\cap T_{s'}\neq\emptyset$ the assertion follows again from Lemma 9 otherwise we obtain a contradiction since Lemma 2.1 of [6] implies that $J=\langle f,s'\rangle$ is free in f and s'. $\square$

Proof of Theorem 4. For isolated subgroups U and V of F_n and F_m an analysis of the proof of Theorem 1 shows that the third case cannot occur since s and f must have a common fixed point if s^{n_1} and f^{n_2} have a common fixed point. This implies that a non-free two-generated subgroup J must lie in $F_n *_U (U\times V)$ or $(U\times V)*_V F_m$ and therefore must be a graph of cyclic groups with underlying graph either a segment or a circle by the arguments above. The isolatedness of U and V however translates into the isolatedness of the edge groups of the induced splitting of J which clearly implies that any (cyclic) vertex group is identical with all its incident edge groups. This implies that J is cyclic in the case of the underlying graph being a segment and the torus group, i.e. free abelian of rank 2, in the case of the underlying graph being a circle. This proves Theorem 4. $\square$

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